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ELECTROMAGNETIC SHOCK WAVES AND THEIR STRUCTURE IN ANISOTROPIC MAGNETIC MATERIALS†

N. I. GVOZDOVSKAYA and A. G. KULIKOVSKII

Moscow

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The analogy between non-linear electromagnetic waves in magnetizable media and non-linear elastic waves in anisotropic media is justified and used. The analogy occurs when dispersion and dissipation are ignored. By using existing results [1] one can therefore immediately formulate all that relates to investigating relations in Riemann waves and electromagnetic shock waves. To describe the structure of electromagnetic shock waves in magnetic materials the Laundau–Lifshits equation is employed, which differs considerably from the relations used to describe the structure of shock waves in elastic media. A consequence of this is that the set of permissible shock waves (i.e. possessing a structure) acquires a complex structure and differs considerably from the analogous set for elastic shock waves. It is shown below, in particular, that the set of permissible electromagnetic shock waves is not the same as the set of initially evolution discontinuities. The requirement that a structure should exist distinguishes, on certain parts of the shock adiabat, a set which is a dashed line with a very short dash length. In addition, there is a large number of individual points on the shock adiabat of the electromagnetic shock waves. Each point corresponds to a discontinuity with a separate velocity of motion, recalling the slow-combustion front in gas dynamics.

1. ELECTROMAGNETIC SHOCK WAVES IN MAGNETIZABLE MEDIA. THE ANALOGY WITH ELASTIC WAVES

Electromagnetic shock waves in media have been investigated in detail (see, for example, [2–4]) in the case of plane-polarized waves, when the magnetic field ahead of and behind the discontinuity and the vector normal to the surface of the discontinuity lie in one plane. In the case of non-electrically conducting unpolarizable media the equations describing continuous one-dimensional waves and the relations on the discontinuities can be written in the form

$$\begin{aligned}
&\frac{\partial \varepsilon_\alpha}{\partial t} - \frac{\partial H_\alpha}{\partial x} = 0, \quad \frac{\partial B_3}{\partial t} = 0, \quad \frac{\partial B_3}{\partial x} = 0 \\
&W[\varepsilon_\alpha] + [H_\alpha] = 0, \quad [B_3] = 0 \\
&\frac{\partial B_\alpha}{\partial t} - \frac{\partial \varepsilon_\alpha}{\partial x} = 0, \quad \frac{\partial E_3}{\partial t} = 0, \quad \frac{\partial E_3}{\partial x} = 0 \\
&W[B_\alpha] + [\varepsilon_\alpha] = 0, \quad [E_3] = 0
\end{aligned} \tag{1.1}$$

The subscript α takes values of 1 or 2 and $\varepsilon_1 = E_2$, $\varepsilon_2 = -E_1$; E_i, H_i, B_i ($i = 1, 2, 3$) are the components of the electric and magnetic field and magnetic induction vectors in a Cartesian system of coordinates. For simplicity, we will assume that the electric induction does not differ from the electric field. The case when the permittivity of the medium is a scalar and constant can easily be reduced to the case considered. The x_3 axis is chosen to be normal to the wave front. In (1.1) and henceforth we will omit the subscript on x_3, i.e. $x_3 \equiv x$. The variables x and t are chosen so that the velocity of light is equal to unity. The square brackets denote a jump in the value at the discontinuity $[B_\alpha] = B_\alpha^+ - B_\alpha^-$, where the minus sign corresponds to the state ahead the discontinuity while the plus sign corresponds to the state behind the discontinuity; W is the velocity of motion of the discontinuity, $W < 1$. For convenience (and to simplify a comparison with elastic waves) we will choose the units of measurement so that the numerical values of the quantities E_i, H_i and B_i are $\sqrt{(4\pi)}$ times greater than in the Gaussian system of units. The energy per unit volume of the medium (the medium is stationary and in a state of thermodynamic equilibrium) and also the expressions for H_α can then be written in the form

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$$U(E_\alpha, B_\alpha, s) = \frac{1}{2}(\varepsilon_1^2 + \varepsilon_2^2) + \Phi(B_\alpha) + T(s - s_0), \quad \alpha = 1, 2 \tag{1.2}$$

$$H_\alpha = \partial\Phi/\partial B_\alpha \tag{1.3}$$

The form of the dependence of U on the entropy s per unit volume of the medium, assumed in (1.2), is due to the fact that, for simplicity, below we will consider processes with fairly small entropy changes. In particular, we will consider electromagnetic shock waves of moderate amplitude. The fact that the change in s is non-negative will be used later as one of the rules for choosing permissible discontinuities.

Equations (1.1)–(1.3) are identical, apart from the notation, with the similar relations for an incompressible homogeneous elastic medium [1, 5]. For complete coincidence it is sufficient to replace ε_α by v_α (the velocity of the medium, $v_3 = 0$), B_α by u_α ($u_\alpha = \partial w_\alpha/\partial x$ is a measure of the deformation of the medium, w_α are the components of the displacement vector and $u_3 = 0$), and H_α by $\sigma_{3\alpha}/\rho_0$ (ρ_0 is the density of the medium and $\sigma_{3\alpha}$ are the components of the stress tensor). In the case of elastic waves U and s must be understood as the energy and entropy per unit mass of the medium.

In many cases the energy of the magnetic field in the medium is specified in a form suitable both for thermodynamically equilibrium states and for non-equilibrium states. It is assumed that this energy is a function of two arguments

$$\Phi_m = \Phi_m(B_i, M_k), \quad M_i = B_i - H_i, \quad i, k = 1, 2, 3 \tag{1.4}$$

The components of the magnetization vector M_i were defined above so that their numerical values are $\sqrt{(4\pi)}$ times smaller than in the Gaussian system of units. The energy of the magnetic field in equilibrium states (in particular, ahead of and behind the discontinuity) has the form

$$\Phi(B_i) = \min_{M_k} \Phi_m(B_i, M_k); \quad i, k = 1, 2, 3 \tag{1.5}$$

while M_k for specified values B_i can be found from the equations

$$\partial\Phi_m/\partial M_k = 0 \tag{1.6}$$

When the magnetic field exceeds the saturation threshold and at sufficiently low temperatures, we can assume that the magnetization vector is constant in modulus $|\mathbf{M}| = M = \text{const}$ [6–8]. Henceforth M_α will be assumed to be independent quantities, while $M_3 = \sqrt{(M^2 - M_1^2 - M_2^2)}$. If the energy of the magnetic field is mainly determined by the mutual orientation of the vectors $\mathbf{B}$ and $\mathbf{M}$ and depends only slightly on the interaction between the vector $\mathbf{M}$ and the medium (this is a typical cases [6, 7]), then, taking the equalities $B_3 = \text{const}$ and $M = \text{const}$ into account, the expression for the internal energy can be written, apart from a constant, in the form [6, 7]

$$\Phi_m(B_\alpha, M_\alpha) = \frac{1}{2}(B_1^2 + B_2^2) - B_\alpha M_\alpha - B_3\sqrt{M^2 - M_1^2 - M_2^2} + g\varphi(M_\alpha) \tag{1.7}$$

The form of the function $\varphi(M_\alpha)$ is determined by the properties of the material. Here g is a small parameter representing the interaction between the medium and the magnetization vector. We will henceforth neglect terms of the order of g^2. The two middle terms on the right-hand side of (1.7) are a scalar product, and for comparison with the standard form of this term we note that $\mathbf{M} \cdot \mathbf{B} = \mathbf{M} \cdot \mathbf{H} + \text{const}$, since $\mathbf{B} = \mathbf{H} + \mathbf{M}$, $M^2 = \text{const}$. In expression (1.7) we have omitted terms which express the dependence of the energy on the spatial derivatives of the vectors characterizing the magnetic field. We will neglect these terms, since we propose below to consider solutions which depend slowly on the coordinates (the characteristic linear dimension of the solution must be much greater than the typical thickness of a domain wall $l_* \approx 10^{-6}$ cm). Taking the first equation of (1.5) into account, it follows from (1.7) and (1.5) that

$$\partial\Phi_m / \partial B_\alpha = \partial\Phi / \partial B_\alpha = B_\alpha - M_\alpha = H_\alpha \tag{1.8}$$

i.e. Eqs (1.3) are satisfied.

Using the fact that g is small and Eq. (1.5), we obtain from (1.7)

$$\Phi(B_\alpha)=\frac{1}{2}(B_1^2+B_2^2)-M\sqrt{B_3^2+B_1^2+B_2^2}+g\varphi^*(B_\alpha) \tag{1.9}$$

Here φ^* is the value of $\varphi(M_\alpha)$ provided that the vectors **M** and **B** are parallel (terms of the order of g^2 and above are ignored).

For small values of B_α the function Φ, given by (1.9), can be expanded in series in powers of B_α

$$\Phi=\frac{1}{2}\left(1-\frac{M}{B_3}\right)(B_1^2+B_2^2)+\frac{M}{8B_3^3}(B_1^2+B_2^2)^2+\frac{g_1}{2}(B_2^2-B_1^2) \tag{1.10}$$

In the last term, which originates from $g\varphi^*$, in view of the fact that g is small we have retained only terms that are quadratic in B_α (linear terms have no effect on the behaviour of the discontinuities and are omitted), the term B_1B_2 is cancelled by the rotation of the system of coordinates, while the term containing $B_1^2+B_2^2$ is dropped as it is small compared with the first term in (1.10).

Materials exist (for example, a ferromagnetic cubic crystal), for which $\varphi^*(B_\alpha)$ has a quadratic form in terms of B_1 and B_2 without assuming that B_α is small [6].

The theory of discontinuities for elastic media has been developed in [1, 9, 10] for the case when Φ has the form of an expansion in powers of its arguments (the case when $\varkappa<0$). Discontinuities have been investigated in [5] for arbitrary functions of Φ and φ (see also [1]).

Here we will present some results which will be necessary later relating to the case when $g\varphi^*(B_\alpha)=g_1(B_2^2-B_1^2)/2$. As we can conclude from [5], these results are independent of whether the first two terms in (1.9) can be represented in the form of expansion (1.10), but do rest considerably on the form of the function $\varphi^*(B_\alpha)$ assumed above. If we fix the values of B_α^-, the states B_α^+ which satisfy the relations on the discontinuity (1.1), give a curve for all possible values of W in the plane B_1B_2, which it is natural to call the shock adiabat. The version that will be of most interest later is the one where the initial point $A(B_1^-, B_2^-)$ is fairly close to the origin of coordinates so that $(B_1^-)^2+(B_2^-)^2<g/\varkappa$. The shock adiabat in this case is shown in Fig. 1.

The mapping of the shock adiabat in the velocity plane (or the "evolution diagram") is shown in Fig. 2.

In Fig. 2 we have plotted the velocity of the discontinuity W along the horizontal axis, and we have shown on this axis the values of c_1^- and c_2^- representing the characteristic velocities of system (1.1) in the state B_1^-, B_2^-. The vertical axis in Fig. 2 serves only for comparing the velocity of the discontinuity W and the characteristic velocities c_1^+ and c_2^+ in states B_1^+, B_2^+. The values of c_1^+ and c_2^+ vary as a function of B_1^+ and B_2^+. Hence, the representation of these quantities and W along the vertical axis is conventional and just characterizes the fact that the inequalities between these three quantities are satisfied. The thick curves in Figs 1 and 2 denote the initially evolutionary sections of the shock adiabat (i.e. evolutionary on the assumption that there are no other relations apart from the initial ones on the discontinuity (1.1)). The entropy non-decreasing condition is satisfied for those parts of the shock adiabat which lie outside the circle with centre at the origin of coordinates passing through the point A in Fig. 1. As can be seen from Fig. 1, this condition is satisfied for all the initially evolutionary discontinuities.

In Figs 1 and 2 we represent by points the states B_1^+, B_2^+ which satisfy the relations on the discontinuity

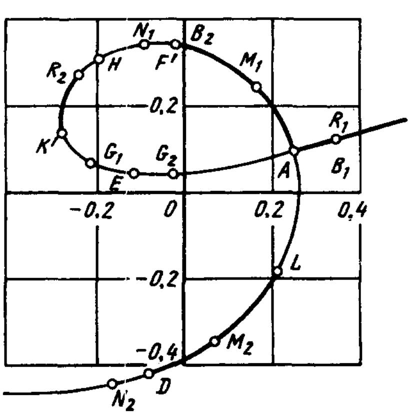

Fig. 1.

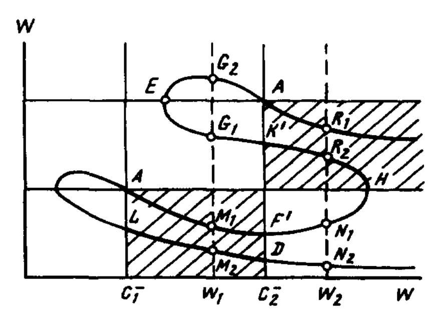

Fig. 2.

for specified B_1^-, B_2^- for two different values of $W = W_1$ and $W = W_2$. The vertical dashed lines in Fig. 2 correspond to these values. The states $B_1^+B_2^+$ corresponding to the value $c_1^- < W_1 < c_2^-$ are shown by the points M_1, M_2, G_1 and G_2 in Fig. 2. The discontinuities $A \to M_1, A \to M_2$ are slow evolutionary discontinuities. The discontinuities $A \to G_1, A \to G_2$ are initially non-evolution discontinuities. The arrangement of the points M_1, M_2, G_1 and G_2 in the B_1B_2 plane is shown in Fig. 1. In exactly the same way the points R_1, R_2, N_1 and N_2 correspond to the states $B_1^+B_2^+$ when $W = W_2 > c_2^-$. The discontinuities $A \to R_1, A \to R_2$ are fast evolutionary discontinuities, while the discontinuities $A \to N_1, A \to N_2$ are non-evolutionary discontinuities. The position of the points R_1, R_2, N_1 and N_2 in the B_1B_2 plane is shown in Fig. 1.

An investigation of the structure of elastic shock waves using the equations of viscoelasticity showed [5, 10, 11], that evolutionary discontinuities and only such possess a structure which enables us to assume them to be "permissible" or physically realizable.

2. THE STRUCTURE OF ELECTROMAGNETIC SHOCK WAVES

The change in the magnetization vector in magnetic materials will be described by the Landau–Lifshits equation [8]

$$\partial \mathbf{M}/\partial t = \gamma(\mathbf{M} \times \mathbf{H}_{ef}) - \lambda \mathbf{H}_{ef} \tag{2.1}$$

The vector $\mathbf{H}_{ef}$ is the gradient of the function $\Phi_m(B_i, M_k)$ with respect to the variable M_k for constant B_α. In the approximations considered here, when $|\mathbf{M}| = M =$ const the gradient is taken on the surface of the sphere $M_1^2 + M_2^2 + M_3^2 = M^2$ and lies in the plane tangent to the sphere. The quantities γ and λ are determined by the properties of the material, and $\lambda/(\gamma M)$ varies in different materials from 10^{-2} to 5×10^{-5}. The last term in the Landau–Lifshits equation is usually written in the form $\lambda\mathbf{M} \times (\mathbf{M} \times \mathbf{H}_{ef})/M^2$, which in the case considered is identical with the one written, since $\mathbf{M} \cdot \mathbf{H}_{ef} = 0$.

The Landau–Lifshits equation has been successfully used to describe fairly rapidly moving waves [6]. There is an experimental confirmation of the Landau–Lifshits attenuation in undeformed magnetic materials [8]. The behaviour of solitons in the isotropic case taking the effective dissipation into account has been investigated in [12], using these equations. A more complex system of equations was used to describe the motion of domain walls or slowly moving waves [7, 13]. Here the dependence of Φ_m on the derivatives of the magnetic field was also taken into account; this leads to the occurrence in Eqs (2.1) of terms with second derivatives $\partial^2 M_\alpha/\partial x^2$. As has already been stated, we must neglect these effects in order that the wavelength l, given by (2.1), should satisfy the condition $l = cW/\gamma H_{ef} \gg l_*$ cm, where cW is the wave velocity. This condition will be assumed to be satisfied.

We choose M_α as the curvilinear coordinates on the sphere $|\mathbf{M}| = M$. Then, using (1.7), and projecting Eq. (2.1) onto the M_1 and M_2 axes, we obtain

$$\frac{\partial M_1}{\partial t} = -\lambda \frac{M^2 - M_1^2 - M_2^2}{M^2 - M_2^2} \left(\frac{\partial \Phi_m}{\partial M_1} \right)_{B_\alpha} - \gamma M_3 \left(\frac{\partial \Phi_m}{\partial M_2} \right)_{B_\alpha} \tag{2.2}$$

$$\frac{\partial M_2}{\partial t} = \gamma M_3 \left(\frac{\partial \Phi_m}{\partial M_1} \right)_{B_\alpha} - \lambda \frac{M^2 - M_1^2 - M_2^2}{M^2 - M_1^2} \left(\frac{\partial \Phi_m}{\partial M_2} \right)_{B_\alpha} \tag{2.3}$$

Multiplying these equations by $(\partial\Phi_m/\partial M_1)_B$ and $(\partial\Phi_m/\partial M_2)_B$ respectively and adding we obtain

$$\left(\frac{\partial \Phi_m}{\partial M_\alpha} \right)_B \frac{\partial M_\alpha}{\partial t} = -\lambda (M^2 - M_1^2 - M_2^2) \left[\frac{1}{M^2 - M_2^2} \left(\frac{\partial \Phi_m}{\partial M_1} \right)_{B_\alpha}^2 + \frac{1}{M^2 - M_1^2} \left(\frac{\partial \Phi_m}{\partial M_2} \right)_{B_\alpha}^2 \right] \leqslant 0 \tag{2.4}$$

As follows from (2.2)–(2.3), inequality (2.4) becomes an equality only when $\partial M_1/\partial t = 0$, $\partial M_2/\partial t = 0$.

We will consider the solution of the equations describing the behaviour of an electromagnetic field in the form of a travelling wave in which all the quantities depend on the variable $\xi = Wt - x$, and we will seek a solution of the problem of the structure of the electromagnetic shock wave. The value $\xi = -\infty$ corresponds to the state ahead of the discontinuity, while the value $\xi = \infty$ corresponds to the state behind the discontinuity, and W is the velocity of the wave in question. The solution is defined by differential equations (1.1) and Eqs (2.2) and (2.3) in which $\partial/\partial t$ must be replaced by $Wd/d\xi$ and $\partial/\partial x$ must be replaced by $-d/d\xi$. Equations (1.1) can then be integrated and, taking into account the equation $B_\alpha = H_\alpha + M_\alpha$ and the conditions for $\xi = -\infty$, give

$$B_\alpha = B_\alpha^- + \frac{1}{1-W^2}(M_\alpha - M_\alpha^-) \tag{2.5}$$

where B_α^- and M_α^- are the values of B_α and M_α when $\xi = -\infty$. This enables us to introduce the function

$$\Phi^*(M_1, M_2) = -\frac{(M_1+P_1)^2+(M_2+P_2)^2}{2(1-W^2)} - B_3\sqrt{M^2-M_1^2-M_2^2} + g\varphi(M_\beta)$$
$$P_\alpha = M_\alpha^- - (1-W^2)B_\alpha^- = M_\alpha^-\left(1-\frac{1-W^2}{M_3^-}B_3\right) \tag{2.6}$$

which is independent of the instantaneous values of B_1 and B_2 such that, taking (2.5) into account, we have

$$\left(\frac{\partial\Phi_m}{\partial M_\alpha}\right)_{B_k} = \frac{\partial\Phi^*(M_\beta)}{\partial M_\alpha} \tag{2.7}$$

Equations (2.7) enable us to represent the equations describing the structure of the electromagnetic shock wave and relation (2.4) in the form

$$W\frac{dM_1}{d\xi} = -\lambda\frac{M^2-M_1^2-M_2^2}{M^2-M_2^2}\frac{\partial\Phi^*}{\partial M_1} - \gamma M_3\frac{\partial\Phi^*}{\partial M_2} \tag{2.8}$$

$$W\frac{dM_2}{d\xi} = \gamma M_3\frac{\partial\Phi^*}{\partial M_1} - \lambda\frac{M^2-M_1^2-M_2^2}{M^2-M_1^2}\frac{\partial\Phi^*}{\partial M_2} \tag{2.9}$$

$$W\frac{d\Phi^*}{d\xi} = -\lambda(M^2-M_1^2-M_2^2)\times$$
$$\times\left[\frac{1}{M^2-M_2^2}\left(\frac{\partial\Phi^*}{\partial M_1}\right)_B^2 + \frac{1}{M^2-M_1^2}\left(\frac{\partial\Phi^*}{\partial M_2}\right)_B^2\right] \leqslant 0 \tag{2.10}$$

States with $\xi = \pm\infty$ can only correspond to singular points of system (2.8), (2.9), since the derivatives of M_α with respect to ξ, by (2.5), are proportional to the derivatives of B_α, and all the derivatives in the solution of the problem of the structure of the shock wave must vanish as $\xi \to \pm\infty$.

The singular points of system (2.8), (2.9) coincide with the critical points of the function $\Phi^*(M_1, M_2)$, in which the partial derivatives of this function are zero. By virtue of Eqs (2.7), at these points the derivative $\partial\Phi_m(M_\alpha, B_\beta)/\partial M_\gamma = 0$ vanish, which, by (1.6), denotes that the singular points of system (2.8), (2.9) correspond to equilibrium states.

Note that if we eliminate the electric field from the relations on the discontinuity (1.1), then, using the equation $M_i = B_i - H$, we can write it in the form

$$B_\alpha^+ = B_\alpha^- + \frac{1}{1-W^2}(M_\alpha^+ - M_\alpha^-) \tag{2.11}$$

Equations (2.11) differ from (2.5) solely in the presence of the superscript + on B_α and M_α. If we make the variables M_α correspond to B_α, as given by (2.5), the transfer from one singular point of system (2.8), (2.9) to another ensures that the relations on electromagnetic shock waves with thermodynamically equilibrium states in front of and behind the discontinuity are satisfied.

Relations (2.11) show that the mutual arrangement of the singular points of system (2.8), (2.9) in the M_1, M_2 plane differs only in scale from the mutual arrangement of the points in the B_1, B_2 plane, characterizing possible states in front of and behind the discontinuity for a specified value of W. This enables us to use the results from Section 1 to determine the mutual arrangement of the singular points.

By inequality (2.10) all the integral curves of Eqs (2.8) and (2.9) in the M_1, M_2 plane intersect the level lines of the function $\Phi^*(M_1, M_2)$ on the side on which this function decreases. Then the angle that the integral curve makes with the level line will be smaller the smaller the ratio of λ to γ. When $\lambda = 0$ the integral curves coincide with the level lines of the function $\Phi^*(M_\beta)$.

Note that when $\lambda = 0$ (i.e. when there is no dissipation) the closed level lines of the function $\Phi^*(M_1, M_2) = \text{const}$ which do not contain critical points correspond to solutions of Eqs (2.8) and (2.9) that are periodic in ξ, i.e. periodic undamped waves. The level lines entering both ends at a singular point correspond to solitary waves. Particular forms of solitary waves were considered in [13] for the case when the direction of the magnetization vector coincides with the axis of easy magnetization when $\xi = \pm\infty$.

The type of singular points of system (2.8), (2.9), by virtue of relation (2.10), is related in an obvious way to the type of corresponding critical points of the function $\Phi^*(M_\alpha)$. If $\Phi^*(M_\alpha)$ has a maximum, then by virtue of (2.10) the whole neighbourhood of this point is filled with integral curves which depart from it as ξ increases. Consequently, the singular point is a node or a focus. For sufficiently small $\lambda/\gamma M$ (it is this case that we are considering) the singular point is a focus. If $\Phi^*(M_\alpha)$ has a minimum, then for small $\lambda/\gamma M$ the singular point is a focus with entering integral curves. In a similar way one can easily obtain that a saddle corresponds to a saddle.

When the parameters on which the function Φ^* depends change (in the case considered, when W changes) the type of critical point in general may not change so long as this point is isolated. Changes may occur when critical points merge. One of the principal coefficients of the quadratic form, representing the function Φ^* in the neighbourhood of each of the critical points, vanishes when they merge, but has different signs at the critical points before merging occurs, whereas (in the general situation) the second coefficient remains non-zero and of the same sign at both points. This denotes that one of the merging points is a saddle while the second is a node. After merging the points may banish and may change places.

When the critical points merge the intensity of the corresponding discontinuity approaches zero, and at the instant when merging occurs the velocity of the discontinuity W is equal to the characteristic velocity, calculated for the state corresponding to this point. A change in the type of the initial point A with specified coordinates M_1^-, M_2^- occurs when, if the point A merges with another singular point, the velocity of the wave W passes through one of the values of the characteristic velocity at the point A. Critical points disappear after merging, for example, at the point H (Fig. 2) when W increases. Then the velocity W is equal to the characteristic velocity behind the discontinuity. For values of W sufficiently close to unity, and for fixed values of M_1^- and M_2^-, such that $(M_1^-)^2 + (M_2^-)^2 < M^2$, it can be seen from (2.6) that the initial point A is a maximum for the function $\Phi^*(M_1, M_2)$.

Hence, and from the previous discussions, it follows that in the M_1, M_2 plane (Fig. 1) for all $W > c_2^-$ the point A is a maximum of the function $\Phi^*(M_1, M_2)$ (an unstable focus for system (2.8), (2.9)), the points R_1 and R_2 are saddles (and are also saddles for system (2.8), (2.9)), while the points N_1 and N_2 are minima of the function $\Phi^*(M_1, M_2)$ (stable foci for the system of differential equations). When $c_1^- < W < c_2^-$, the point A is a saddle for the function $\Phi^*(M_1, M_2)$ (and for system (2.8), (2.9)), the point G_1 is a saddle for $\Phi^*(M_1, M_2)$ (and a saddle for the system), and the point G_2 is a maximum for $\Phi^*(M_1, M_2)$ (an unstable focus for the system). This enables us to represent the level lines of the function $\Phi^*(M_1, M_2)$ and the integral curves of system (2.8), (2.9) qualitatively.

We will first consider the case when $W > c_2^-$. The position of the singular points (compare with Fig. 1) and the level lines of the function $\Phi^*(M_1, M_2)$ are shown in Fig. 3. The qualitative investigation of the level lines is confirmed by a numerical calculation, one of the results of which is shown in Fig. 3.

It can be seen from Fig. 3 that there is always an integral curve from point A to point R_1. In other words, all the discontinuities corresponding to the branches of the shock adiabat which go from the upper point A to the right in Fig. 2 and to the right upwards in Fig. 1 are permissible.

The integral curves which go from point A to point R_2 accumulate in a narrow strip whose width is smaller the smaller the value of $\lambda/\gamma M$ ($\lambda/\gamma M$ is of the order of magnitude of the angle which the integral curve makes with the level lines of the of the function $\Phi^*(M_1, M_2)$) and the smaller the length of the loop of the level line which passes through the point R_1 and surrounds the point A. This strip, shown hatched in Fig. 3, almost follows the level lines of the of function $\Phi^*(M_1, M_2)$ for small $\lambda/\gamma M$, advancing very slightly in the direction of which Φ^* decreases as ξ increases and describes something like a spiral. There are integral curves between the coils of this spiral which fall within the region considered through the second external loop of the level line passing through the point R_1. The ratio of the width of the strip to the distance between neighbouring strips is of the same order of magnitude as the ratio of the lengths of the loops which compromise the level line, passing through the point R_1. The strip emerging from point A may end at one of the points N_1 or N_2, where the function $\Phi^*(M_1, M_2)$ has a minimum. It may also divide, and part of the integral curves will end at N_1 and part at N_2. Here one of the integral

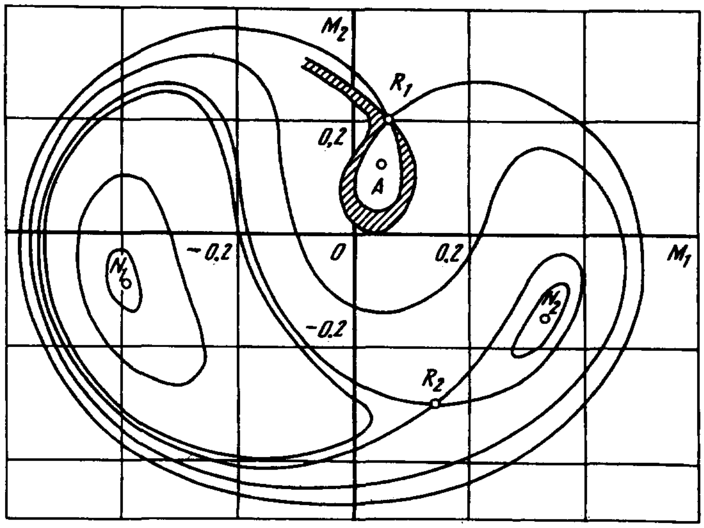

Fig. 3.

curves arrives at the point R_2. In the latter case an evolutionary shock wave $A \to R_2$ will be permissible.

When the parameters characterizing the shock waves change, and in particular when W changes, the level lines of the function $\Phi^*(M_1, M_2)$ will change and therefore the possibilities described above of the integral curves emerging from the point A terminating will alternate. For small $\lambda/\gamma M$ a small change in the level lines due to a change in W is sufficient for the strip consisting of the integral curves emerging from the point A to make several excessive rotations. If the width of the strip is less than the distance between the strips then, for a change in W, the point A will be connected to the point R_2 and then not connected. The change in W required for this will be smaller the smaller the value of $\lambda/\gamma M$.

Hence, the set of permissible discontinuities in the section of the shock adiabat $K'H$ (Figs 1 and 2) recalls the dashed line where the length of the dash is shorter the smaller the value of $\lambda/(\gamma M)$. As regards the ratio of the width of the strip to the distance between neighbouring strips, it seems to be always less than unity and is small when the points A and R_1 are close to one another, i.e. when the velocity W is close to c^-_2. The shock waves $A \to N_1$ and $A \to N_2$ are non-evolutionary due to the fact that there are more boundary conditions on the discontinuity (1.1) than necessary for their evolutionarity. Consequently, these shock-waves are of no interest as they have a tendency to decay when they interact with small perturbations.

We will now consider the case when $c_1^- < W < c_2^-$, assuming initially that the difference $c_2^- - W$ is small. Then the pattern of level lines and integral curves shown in Fig. 3 does not change qualitatively with the exception of a change in the name of the singular points (see Fig. 4).

The initial point A is now a saddle. The separatrices which emerge from this point when ξ increases are shown by the dashed lines in Fig. 4. These separatrices may terminate at the points M_1 and M_2 as

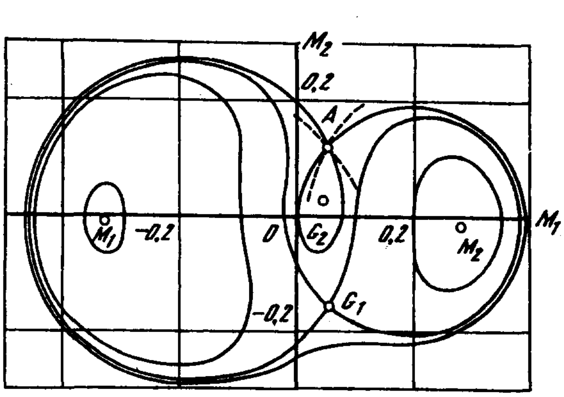

Fig. 4.

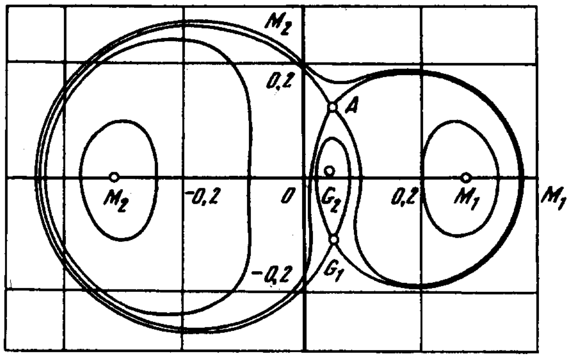

Fig. 5.

$\xi \to \infty$ and may represent the structure of slow evolutionary shock waves. But for a certain special combination of parameters (which must be regarded as an additional relation on the discontinuity) the separatrices may terminate at the point G_1, representing the structure of the shock wave $A \to G_1$ (which, without this additional relation, would be non-evolutionary). If we assume that B_1^- and B_2^- are constants and vary W, then, like the previous case, the versions of the termination of the integral curves emerging from the point A will be changed more often the smaller the value of $\lambda/\gamma M$.

It can be seen from Fig. 4 that the integral curves emerging from the point A, bound a strip composed of the integral curves emerging from the point G_2. If the width of this strip is less than the distance between the strips (which seems to be always true), when W changes there will be intervals when both integral curves emerging from the point A terminate at one of the singular points M_1 or M_2. This denotes that there is no structure in one of the shock waves. On the sections of the shock adiabat AF' and LD, corresponding to slow shock waves, a set of permissible discontinuities is represented in this case by the dashed lines (the dashes are arranged so that for all W at least one of the slow shock waves has a structure). Discontinuities of the type $A \to G_1$, which are only evolutionary by virtue of the above-mentioned specification of the velocity, which is an additional relation on the discontinuity, correspond to the velocities which correspond to the ends of the dashes.

If $c_1^- < W < c_2^-$, but the value of W is close to c_1^-, the pattern of level lines has the form shown in Fig. 5 (when W changes from c_2^- to c_1^- a rearrangement of the level lines occurs; for brevity, intermediate versions will not be considered). In this case a structure of both slow evolutionary shock waves $A \to M_1, A \to M_2$ and none others always exists. Hence, the sections of the shock adiabat corresponding to the slow shock waves for W close to c_1^-, belong as a whole to the set of permissible discontinuities.

When $W < c_1^-$ there are no permissible discontinuities since the initial point A is a stable focus.

In conclusion we note that such an intricately constructed shock adiabat gives rise to problems related to the structure of the solutions of the initial boundary-value problems. In addition, the possibility arises of considerably influencing the structure of the solution by means of a small change in the parameters (for example, an externally applied magnetic field), which occur in the formulation of the problem. From the point of view of the theory of differential equations, the problem considered is one more example of the fact that hyperbolic systems of equations often cannot be used to find the solutions as a whole, since the set of permissible discontinuities is not defined by hyperbolic systems. We note also that the set of permissible discontinuities and the solution of the problem of the structure are qualitatively the same here as in [14], where a simpler example of a partial differential equation without specific physical applications was considered.

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